

# ON THE EXISTENCE OF THE DYNAMICS FOR ANHARMONIC QUANTUM OSCILLATOR SYSTEMS

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**ABSTRACT.** We construct a  $W^*$ -dynamical system describing the dynamics of a class of anharmonic quantum oscillator lattice systems in the thermodynamic limit. Our approach is based on recently proved Lieb-Robinson bounds for such systems on finite lattices [19].

## 1. INTRODUCTION

The dynamics of a finite quantum system, i.e., one with a finite number of degrees of freedom described by a Hilbert space  $\mathcal{H}$ , is given by the Schrödinger equation. The Hamiltonian  $H$  is a densely defined self-adjoint operator on  $\mathcal{H}$ , and for a vector  $\psi(t)$  in the domain of  $H$  the state at time  $t$  satisfies

$$(1.1) \quad i\partial_t \psi(t) = H\psi(t).$$

For all initial conditions  $\psi(0) \in \mathcal{H}$ , the unique solution is given by

$$\psi(t) = e^{-itH}\psi(0), \text{ for all } t \in \mathbb{R}.$$

Due to Stone's Theorem  $e^{-itH}$  is a strongly continuous one-parameter group of unitary operators on  $\mathcal{H}$ , and the self-adjointness of  $H$  is the necessary and sufficient condition for the existence of a unique continuous solution for all times.

An alternative description of this dynamics is the so-called Heisenberg picture in which the time evolution is defined on the algebra of observables instead of the Hilbert space of states. The corresponding Heisenberg equation is

$$(1.2) \quad \partial_t A(t) = i[H, A(t)],$$

where, for each  $t \in \mathbb{R}$ ,  $A(t) \in \mathcal{B}(\mathcal{H})$  is a bounded linear operator on  $\mathcal{H}$ . Its solutions are given by a one-parameter group of  $*$ -automorphisms,  $\tau_t$ , of  $\mathcal{B}(\mathcal{H})$ :

$$A(t) = \tau_t(A(0)).$$

For the description of physical systems we expect the Hamiltonian,  $H$ , to have some additional properties. E.g., for finite systems such as atoms or molecules, stability of the system requires that  $H$  is bounded from below. In this case, the infimum of the spectrum is expected to be an eigenvalue and is called the ground state energy. When the model Hamiltonian,  $H$ , is describing bulk matter rather than finite systems, we expect some additional properties. E.g., the stability of matter requires that the ground state energy has a lower bound proportional to  $N$ , where  $N$  is the number of degree of freedom. Much progress on this stability property has been made in the last several decades [24, 12]. We also expect that the dynamics of local observables of bulk matter, or large systems in general, depends only on the local environment. Mathematically this is best expressed by the existence of the dynamics in the thermodynamic limit, i.e., in infinite volume. This is the question we address in this paper.

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There are two settings that allow one to prove a rich set of important physical properties of quantum dynamical systems, including infinite ones: the  $C^*$  dynamical systems and the  $W^*$  dynamical systems [3]. In both cases, the algebra of observables can be thought of as a norm-closed  $*$ -subalgebra  $\mathcal{A}$  of some algebra of the form  $\mathcal{B}(\mathcal{H})$ , but in the case of the  $W^*$ -dynamical systems we additionally require that the algebra is closed for the weak operator topology, which makes it a von Neumann algebra. For a  $C^*$ -dynamical system the group of automorphisms  $\tau_t$  is assumed to be strongly continuous, i.e., for all  $A \in \mathcal{A}$ , the map  $t \mapsto \tau_t(A)$  is continuous in  $t$  for the operator norm ( $C^*$ -norm) on  $\mathcal{A}$ . In a  $W^*$ -dynamical system the continuity is with respect to the weak topology.

In the case of lattice systems with a finite-dimensional Hilbert space of states associated with each lattice sites, such as quantum spin-lattice systems and lattice fermions, it has been known for a long time that under rather general conditions the dynamics can be described by a  $C^*$  dynamical system, including in the thermodynamic limit [4]. When the Hilbert space at each site is infinite-dimensional and the finite-system Hamiltonians are unbounded, this is no longer possible and the *weak continuity* becomes a natural assumption.

The class of systems we will primarily focus on here are lattices of quantum oscillators but the underlying lattice structure is not essential for our method. Systems defined on suitable graphs, such as the systems considered in [6, 7] can also be analyzed with the same methods. In a recent preprint [1], it was shown that convergence of the dynamics in the thermodynamic limit can be obtained for a modified topology. Here, we follow a somewhat different approach. The main difference is that we study the thermodynamic limit of anharmonic perturbations of an *infinite* harmonic lattice system described by an explicit  $W^*$ -dynamical system. The more traditional way is to first define the dynamics of anharmonic systems in finite volume (which can be done by standard means [21]), and then to study the limit in which the volume tends to infinity. This is what is done in [1], but it appears that controlling the continuity of the limiting dynamics is more straightforward in our approach. In fact, we are able to show that the resulting dynamics for the class of anharmonic lattices we study is indeed weakly continuous, and we obtain a  $W^*$ -dynamical system for the infinite system. The  $W^*$ -dynamical setting is obtained by considering the GNS representation of a ground state or thermal equilibrium state of the harmonic system. The ground states and thermal states are quasi-free states in the sense of [22], or convex mixtures of quasi-free states. In the ground state case the GNS representations are the well-known Fock representations. For the thermal states the GNS representations have been constructed by Araki and Woods [2].

Common to both approaches, ours and the one of [1], is the crucial role played by an estimate of the speed of propagation of perturbations in the system, commonly referred to as Lieb-Robinson bounds [8, 11, 16, 17, 18]. Briefly, if  $A$  and  $B$  are two observables of a spatially extended system, localized in regions  $X$  and  $Y$  of our graph, respectively, and  $\tau_t$  denotes the time evolution of the system then, a Lieb-Robinson bound is an estimate of the form

$$\|[\tau_t(A), B]\| \leq C e^{-a(d(X,Y)-v|t|)},$$

where  $C, a$ , and  $v$  are positive constants and  $d(X, Y)$  denotes the distance between  $X$  and  $Y$ . Lieb-Robinson bounds for anharmonic lattice systems were recently proved in [19], and this work builds on the results obtained there. Our results are mainly limited to short-range interactions that are either bounded or unbounded perturbations of the harmonic interaction (linear springs).

To conclude the introduction, let us mention that the same questions, the existence of the dynamics for infinite oscillator lattices, can and has been asked for classical systems. Two classic papers are [10, 15]. Many properties of this classical infinite volume harmonic dynamics have been studied in detail e.g. [23, 9] and some recent progress on locality estimates for anharmonic systems is reported in [5, 20].

The paper is organized as follows. We begin with a section discussing bounded interactions. In this case, the existence of the dynamics follows by mimicking the proof valid in the context of quantum spins systems. Section 3 describes the infinite volume harmonic dynamics on general

graphs. It is motivated by an explicit example on  $\mathbb{Z}^d$ . Next, in Section 4, we discuss finite volume perturbations of the infinite volume harmonic dynamics and prove that such systems satisfy a Lieb-Robinson bound. In Section 5 we demonstrate that the existence of the dynamics and its continuity follow from the Lieb-Robinson estimates established in the previous section.

## 2. BOUNDED INTERACTIONS

The goal of this section is to prove the existence of the dynamics for oscillator systems with bounded interactions. Since oscillator systems with bounded interactions can be treated as a special case of more general models with bounded interactions, we will use a slightly more general setup in this section, which we now introduce.

We will denote by  $\Gamma$  the underlying structure on which our models will be defined. Here  $\Gamma$  will be an arbitrary set of sites equipped with a metric  $d$ . For  $\Gamma$  with countably infinite cardinality, we will need to assume that there exists a non-increasing function  $F : [0, \infty) \rightarrow (0, \infty)$  for which:

i)  $F$  is uniformly integrable over  $\Gamma$ , i.e.,

$$(2.1) \quad \|F\| := \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x, y)) < \infty,$$

and

ii)  $F$  satisfies

$$(2.2) \quad C := \sup_{x, y \in \Gamma} \sum_{z \in \Gamma} \frac{F(d(x, z)) F(d(z, y))}{F(d(x, y))} < \infty.$$

Given such a set  $\Gamma$  and a function  $F$ , by the triangle inequality, for any  $a \geq 0$  the function

$$F_a(x) = e^{-ax} F(x),$$

also satisfies i) and ii) above with  $\|F_a\| \leq \|F\|$  and  $C_a \leq C$ .

In typical examples, one has that  $\Gamma \subset \mathbb{Z}^d$  for some integer  $d \geq 1$ , and the metric is just given by  $d(x, y) = |x - y| = \sum_{j=1}^d |x_j - y_j|$ . In this case, the function  $F$  can be chosen as  $F(|x|) = (1 + |x|)^{-d-\epsilon}$  for any  $\epsilon > 0$ .

To each  $x \in \Gamma$ , we will associate a Hilbert space  $\mathcal{H}_x$ . In many relevant systems, one considers  $\mathcal{H}_x = L^2(\mathbb{R}, dq_x)$ , but this is not essential. With any finite subset  $\Lambda \subset \Gamma$ , the Hilbert space of states over  $\Lambda$  is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x,$$

and the local algebra of observables over  $\Lambda$  is then defined to be

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x),$$

where  $\mathcal{B}(\mathcal{H}_x)$  denotes the algebra of bounded linear operators on  $\mathcal{H}_x$ .

If  $\Lambda_1 \subset \Lambda_2$ , then there is a natural way of identifying  $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$ , and we may thereby define the algebra of quasi-local observables by the inductive limit

$$\mathcal{A}_\Gamma = \bigcup_{\Lambda \subset \Gamma} \mathcal{A}_\Lambda,$$

where the union is over all finite subsets  $\Lambda \subset \Gamma$ ; see [3, 4] for a discussion of these issues in general.

The result discussed in this section corresponds to bounded perturbations of local self-adjoint Hamiltonians. We fix a collection of on-site local operators  $H^{\text{loc}} = \{H_x\}_{x \in \Gamma}$  where each  $H_x$  is a self-adjoint operator over  $\mathcal{H}_x$ . In addition, we will consider a general class of bounded perturbations. These are defined in terms of an interaction  $\Phi$ , which is a map from the set of subsets of  $\Gamma$  to  $\mathcal{A}_\Gamma$  with the property that for each finite set  $X \subset \Gamma$ ,  $\Phi(X) \in \mathcal{A}_X$  and  $\Phi(X)^* = \Phi(X)$ . As with the

Lieb-Robinson bound proven in [19], we will need a growth condition on the set of interactions  $\Phi$  for which we can prove the existence of the dynamics in the thermodynamic limit. This condition is expressed in terms of the following norm. For any  $a \geq 0$ , denote by  $\mathcal{B}_a(\Gamma)$  the set of interactions for which

$$(2.3) \quad \|\Phi\|_a := \sup_{x,y \in \Gamma} \frac{1}{F_a(d(x,y))} \sum_{X \ni x,y} \|\Phi(X)\| < \infty.$$

Now, for a fixed sequence of local Hamiltonians  $H^{\text{loc}} = \{H_x\}_{x \in \Gamma}$ , as described above, an interaction  $\Phi \in \mathcal{B}_a(\Gamma)$ , and a finite subset  $\Lambda \subset \Gamma$ , we will consider self-adjoint Hamiltonians of the form

$$(2.4) \quad H_\Lambda = H_\Lambda^{\text{loc}} + H_\Lambda^\Phi = \sum_{x \in \Lambda} H_x + \sum_{X \subset \Lambda} \Phi(X),$$

acting on  $\mathcal{H}_\Lambda$  (with domain given by  $\bigotimes_{x \in \Lambda} D(H_x)$  where  $D(H_x) \subset \mathcal{H}_x$  denotes the domain of  $H_x$ ). As these operators are self-adjoint, they generate a dynamics, or time evolution,  $\{\tau_t^\Lambda\}$ , which is the one parameter group of automorphisms defined by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \quad \text{for any } A \in \mathcal{A}_\Lambda.$$

**Theorem 2.1.** *Under the conditions stated above, for all  $t \in \mathbb{R}$ ,  $A \in \mathcal{A}_\Gamma$ , the norm limit*

$$(2.5) \quad \lim_{\Lambda \rightarrow \Gamma} \tau_t^\Lambda(A) = \tau_t(A)$$

*exists in the sense of non-decreasing exhaustive sequences of finite volumes  $\Lambda$  and defines a group of  $*$ -automorphisms  $\tau_t$  on the completion of  $\mathcal{A}_\Gamma$ . The convergence is uniform for  $t$  in a compact set.*

*Proof.* Let  $\Lambda \subset \Gamma$  be a finite set. Consider the unitary propagator

$$(2.6) \quad \mathcal{U}_\Lambda(t, s) = e^{itH_\Lambda^{\text{loc}}} e^{-i(t-s)H_\Lambda} e^{-isH_\Lambda^{\text{loc}}}$$

and its associated *interaction-picture* evolution defined by

$$(2.7) \quad \tau_{t,\text{int}}^\Lambda(A) = \mathcal{U}_\Lambda(0, t) A \mathcal{U}_\Lambda(t, 0) \quad \text{for all } A \in \mathcal{A}_\Gamma.$$

Clearly,  $\mathcal{U}_\Lambda(t, t) = \mathbb{1}$  for all  $t \in \mathbb{R}$ , and it is also easy to check that

$$i \frac{d}{dt} \mathcal{U}_\Lambda(t, s) = H_\Lambda^{\text{int}}(t) \mathcal{U}_\Lambda(t, s) \quad \text{and} \quad -i \frac{d}{ds} \mathcal{U}_\Lambda(t, s) = \mathcal{U}_\Lambda(t, s) H_\Lambda^{\text{int}}(s)$$

with the time-dependent generator

$$(2.8) \quad H_\Lambda^{\text{int}}(t) = e^{itH_\Lambda^{\text{loc}}} H_\Lambda^\Phi e^{-itH_\Lambda^{\text{loc}}} = \sum_{Z \subset \Lambda} e^{itH_\Lambda^{\text{loc}}} \Phi(Z) e^{-itH_\Lambda^{\text{loc}}}.$$

Fix  $T > 0$  and  $X \subset \Gamma$  finite. For any  $A \in \mathcal{A}_X$ , we will show that for any non-decreasing, exhausting sequence  $\{\Lambda_n\}$  of  $\Gamma$ , the sequence  $\{\tau_{t,\text{int}}^{\Lambda_n}(A)\}$  is Cauchy in norm, uniformly for  $t \in [-T, T]$ . Moreover, the bounds establishing the Cauchy property depend on  $A$  only through  $X$  and  $\|A\|$ . Since

$$\tau_t^\Lambda(A) = \tau_{t,\text{int}}^\Lambda \left( e^{itH_\Lambda^{\text{loc}}} A e^{-itH_\Lambda^{\text{loc}}} \right) = \tau_{t,\text{int}}^\Lambda \left( e^{it \sum_{x \in X} H_x} A e^{-it \sum_{x \in X} H_x} \right),$$

an analogous statement then immediately follows for  $\{\tau_t^{\Lambda_n}(A)\}$ , since they are all also localized in  $X$  and have the same norm as  $\|A\|$ .

Take  $n \leq m$  with  $X \subset \Lambda_n \subset \Lambda_m$  and calculate

$$(2.9) \quad \tau_{t,\text{int}}^{\Lambda_m}(A) - \tau_{t,\text{int}}^{\Lambda_n}(A) = \int_0^t \frac{d}{ds} \{ \mathcal{U}_{\Lambda_m}(0, s) \mathcal{U}_{\Lambda_n}(s, t) A \mathcal{U}_{\Lambda_n}(t, s) \mathcal{U}_{\Lambda_m}(s, 0) \} ds.$$

A short calculation shows that

$$\begin{aligned}
(2.10) \quad & \frac{d}{ds} \mathcal{U}_{\Lambda_m}(0, s) \mathcal{U}_{\Lambda_n}(s, t) A \mathcal{U}_{\Lambda_n}(t, s) \mathcal{U}_{\Lambda_m}(s, 0) \\
&= i \mathcal{U}_{\Lambda_m}(0, s) \left[ (H_{\Lambda_m}^{\text{int}}(s) - H_{\Lambda_n}^{\text{int}}(s)), \mathcal{U}_{\Lambda_n}(s, t) A \mathcal{U}_{\Lambda_n}(t, s) \right] \mathcal{U}_{\Lambda_m}(s, 0) \\
&= i \mathcal{U}_{\Lambda_m}(0, s) e^{isH_{\Lambda_n}^{\text{loc}}} \left[ \tilde{B}(s), \tau_{s-t}^{\Lambda_n}(\tilde{A}(t)) \right] e^{-isH_{\Lambda_n}^{\text{loc}}} \mathcal{U}_{\Lambda_m}(s, 0),
\end{aligned}$$

where

$$(2.11) \quad \tilde{A}(t) = e^{-itH_{\Lambda_n}^{\text{loc}}} A e^{itH_{\Lambda_n}^{\text{loc}}} = e^{-itH_X^{\text{loc}}} A e^{itH_X^{\text{loc}}}$$

and

$$\begin{aligned}
(2.12) \quad \tilde{B}(s) &= e^{-isH_{\Lambda_n}^{\text{loc}}} (H_{\Lambda_m}^{\text{int}}(s) - H_{\Lambda_n}^{\text{int}}(s)) e^{isH_{\Lambda_n}^{\text{loc}}} \\
&= \sum_{Z \subset \Lambda_m} e^{isH_{\Lambda_m \setminus \Lambda_n}^{\text{loc}}} \Phi(Z) e^{-isH_{\Lambda_m \setminus \Lambda_n}^{\text{loc}}} - \sum_{Z \subset \Lambda_n} \Phi(Z) \\
&= \sum_{\substack{Z \subset \Lambda_m: \\ Z \cap \Lambda_m \setminus \Lambda_n \neq \emptyset}} e^{isH_{\Lambda_m \setminus \Lambda_n}^{\text{loc}}} \Phi(Z) e^{-isH_{\Lambda_m \setminus \Lambda_n}^{\text{loc}}}
\end{aligned}$$

Combining the results of (2.9) -(2.12), and using unitarity, we find that

$$(2.13) \quad \left\| \tau_{t, \text{int}}^{\Lambda_m}(A) - \tau_{t, \text{int}}^{\Lambda_n}(A) \right\| \leq \int_0^t \left\| \left[ \tau_{s-t}^{\Lambda_n}(\tilde{A}(t)), \tilde{B}(s) \right] \right\| ds$$

and by the Lieb-Robinson bound proven in [19], it is clear that

$$\begin{aligned}
(2.14) \quad & \left\| \left[ \tau_{s-t}^{\Lambda_n}(\tilde{A}(t)), \tilde{B}(s) \right] \right\| \\
&\leq \sum_{\substack{Z \subset \Lambda_m: \\ Z \cap \Lambda_m \setminus \Lambda_n \neq \emptyset}} \left\| \left[ \tau_{s-t}^{\Lambda_n}(\tilde{A}(t)), e^{isH_{\Lambda_m \setminus \Lambda_n}^{\text{loc}}} \Phi(Z) e^{-isH_{\Lambda_m \setminus \Lambda_n}^{\text{loc}}} \right] \right\| \\
&\leq \frac{2\|A\|}{C_a} \left( e^{2\|\Phi\|_a C_a |t-s|} - 1 \right) \sum_{y \in \Lambda_m \setminus \Lambda_n} \sum_{\substack{Z \subset \Lambda_m: \\ y \in Z}} \|\Phi(Z)\| \sum_{x \in X} \sum_{z \in Z} F_a(d(x, z)) \\
&\leq \frac{2\|A\|}{C_a} \left( e^{2\|\Phi\|_a C_a |t-s|} - 1 \right) \sum_{y \in \Lambda_m \setminus \Lambda_n} \sum_{z \in \Lambda_m} \sum_{\substack{Z \subset \Lambda_m: \\ y, z \in Z}} \|\Phi(Z)\| \sum_{x \in X} F_a(d(x, z)) \\
&\leq \frac{2\|A\| \|\Phi\|_a}{C_a} \left( e^{2\|\Phi\|_a C_a |t-s|} - 1 \right) \sum_{y \in \Lambda_m \setminus \Lambda_n} \sum_{x \in X} \sum_{z \in \Lambda_m} F_a(d(x, z)) F_a(d(z, y)) \\
&\leq 2\|A\| \|\Phi\|_a \left( e^{2\|\Phi\|_a C_a |t-s|} - 1 \right) \sum_{y \in \Lambda_m \setminus \Lambda_n} \sum_{x \in X} F_a(d(x, y)).
\end{aligned}$$

With the estimate above and the properties of the function  $F_a$ , it is clear that

$$(2.15) \quad \sup_{t \in [-T, T]} \left\| \tau_{t, \text{int}}^{\Lambda_m}(A) - \tau_{t, \text{int}}^{\Lambda_n}(A) \right\| \rightarrow 0 \quad \text{as } n, m \rightarrow \infty,$$

and the rate of convergence only depends on the norm  $\|A\|$  and the set  $X$  where  $A$  is supported. This proves the claim.  $\square$

If all local Hamiltonians  $H_x$  are bounded,  $\{\tau_t\}$  is strongly continuous. If the  $H_x$  are allowed to be densely defined unbounded self-adjoint operators, we only have weak continuity and the dynamics is more naturally defined on a von Neumann algebra. This can be done when we have a sufficiently nice invariant state for the model with only the on-site Hamiltonians. E.g., suppose that for each

$x \in \Gamma$ , we have a normalized eigenvector  $\phi_x$  of  $H_x$ . Then, for all  $A \in \mathcal{A}_\Lambda$ , for any finite  $\Lambda \subset \Gamma$ , define

$$(2.16) \quad \rho(A) = \left\langle \bigotimes_{x \in \Lambda} \phi_x, A \bigotimes_{x \in \Lambda} \phi_x \right\rangle.$$

$\rho$  can be regarded as a state of the infinite system defined on the norm completion of  $\mathcal{A}_\Gamma$ . The GNS Hilbert space  $\mathcal{H}_\rho$  of  $\rho$  can be constructed as the closure of  $\mathcal{A}_\Gamma \bigotimes_{x \in \Gamma} \phi_x$ . Let  $\psi \in \mathcal{A}_\Gamma \bigotimes_{x \in \Gamma} \phi_x$ . Then

$$(2.17) \quad \begin{aligned} \|(\tau_t(A) - \tau_{t_0}(A))\psi\| &\leq \left\| \left( \tau_t(A) - \tau_t^{(\Lambda_n)}(A) \right) \psi \right\| \\ &+ \left\| \left( \tau_t^{(\Lambda_n)}(A) - \tau_{t_0}^{(\Lambda_n)}(A) \right) \psi \right\| + \left\| \left( \tau_{t_0}^{(\Lambda_n)}(A) - \tau_{t_0}(A) \right) \psi \right\|, \end{aligned}$$

For sufficiently large  $\Lambda_n$ , the  $\lim_{t \rightarrow t_0}$  of middle term vanishes by Stone's theorem. The two other terms are handled by 2.5. It is clear how to extend the continuity to  $\psi \in \mathcal{H}_\rho$ .

We will discuss this type of situation in more detail in the next three sections where we consider models that include quadratic (unbounded) interactions as well.

### 3. THE HARMONIC LATTICE

As noted in the introduction, we will consider anharmonic perturbations of infinite harmonic lattices. In this section we discuss the properties of the harmonic systems that we need to assume in general in order to study the perturbations in the thermodynamic limit. We will also show in detail that a standard harmonic lattice model possesses all the required properties.

**3.1. The CCR algebra of observables.** We begin by introducing the CCR algebra on which the harmonic dynamics will be defined. Following [14], one can define the CCR algebra over any real linear space  $\mathcal{D}$  equipped with a non-degenerate, symplectic bilinear form  $\sigma$ , i.e.  $\sigma : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{R}$  with the property that if  $\sigma(f, g) = 0$  for all  $f \in \mathcal{D}$ , then  $g = 0$ , and

$$(3.1) \quad \sigma(f, g) = -\sigma(g, f) \quad \text{for all } f, g \in \mathcal{D}.$$

In typical examples,  $\mathcal{D}$  will be a complex inner product space associated with  $\Gamma$ , e.g.  $\mathcal{D} = \ell^2(\Gamma)$  or a subspace thereof such as  $\mathcal{D} = \ell^1(\Gamma)$ , or  $\ell^2(\Gamma_0)$ , with  $\Gamma_0 \subset \Gamma$ , and

$$(3.2) \quad \sigma(f, g) = \text{Im} [\langle f, g \rangle].$$

The Weyl operators over  $\mathcal{D}$  are defined by associating non-zero elements  $W(f)$  to each  $f \in \mathcal{D}$  which satisfy

$$(3.3) \quad W(f)^* = W(-f) \quad \text{for each } f \in \mathcal{D},$$

and

$$(3.4) \quad W(f)W(g) = e^{-i\sigma(f, g)/2} W(f + g) \quad \text{for all } f, g \in \mathcal{D}.$$

It is well-known that there is a unique, up to  $*$ -isomorphism,  $C^*$ -algebra generated by these Weyl operators with the property that  $W(0) = \mathbb{1}$ ,  $W(f)$  is unitary for all  $f \in \mathcal{D}$ , and  $\|W(f) - \mathbb{1}\| = 2$  for all  $f \in \mathcal{D} \setminus \{0\}$ , see e.g. Theorem 5.2.8 [4]. This algebra, commonly known as the CCR algebra, or Weyl algebra, over  $\mathcal{D}$ , we will denote by  $\mathcal{W} = \mathcal{W}(\mathcal{D})$ .

**3.2. Quasi-free dynamics.** The anharmonic dynamics we study in this paper will be defined as perturbations of harmonic, technically *quasi-free*, dynamics. A quasi-free dynamics on  $\mathcal{W}(\mathcal{D})$  is a one-parameter group of  $*$ -automorphisms  $\tau_t$  of the form

$$(3.5) \quad \tau_t(W(f)) = W(T_t f), \quad f \in \mathcal{D}$$

where  $T_t : \mathcal{D} \rightarrow \mathcal{D}$  is a group of real-linear, symplectic transformations, i.e.,

$$(3.6) \quad \sigma(T_t f, T_t g) = \sigma(f, g).$$

As  $\|W(f) - W(g)\| = 2$  for all  $f \neq g \in \mathcal{D}$ , one should not expect  $\tau_t$  to be strongly continuous; only a weaker form of continuity is present. This means that  $\tau_t$  does *not* define a  $C^*$ -dynamical system on  $\mathcal{W}$ , and thus we look for a  $W^*$ -dynamical setting in which the weaker form of continuity is naturally expressed.

In the present context, it suffices to regard a  $W^*$ -dynamical system as a pair  $\{\mathcal{M}, \alpha_t\}$  where  $\mathcal{M}$  is a von Neumann algebra and  $\alpha_t$  is a weakly continuous, one parameter group of  $*$ -automorphisms of  $\mathcal{M}$ . For the harmonic systems we are considering, a specific  $W^*$ -dynamical system arises as follows. Let  $\rho$  be a state on  $\mathcal{W}$  and denote by  $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$  the corresponding GNS representation. We will assume that  $\rho$  is both regular and  $\tau_t$ -invariant. Recall that  $\rho$  is regular if and only if  $t \mapsto \rho(W(tf))$  is continuous for all  $f \in \mathcal{D}$ , and  $\tau_t$ -invariance means

$$(3.7) \quad \rho(\tau_t(A)) = \rho(A) \quad \text{for all } A \in \mathcal{W}.$$

For the von Neumann algebra  $\mathcal{M}$ , take the weak-closure of  $\pi_\rho(\mathcal{W})$  in  $\mathcal{L}(\mathcal{H}_\rho)$  and let  $\alpha_t$  be the weakly continuous, one parameter group of  $*$ -automorphisms of  $\mathcal{M}$  obtained by lifting  $\tau_t$  to  $\mathcal{M}$ . The latter step is possible since  $\rho$  is  $\tau_t$ -invariant, see e.g. Corollary 2.3.17 [3].

**3.3. Lieb-Robinson bounds for harmonic lattices.** To prove the existence of the dynamics for anharmonic models, we use that the unperturbed harmonic system satisfies a Lieb-Robinson bound. Such an estimate depends directly on properties of  $\sigma$  and  $T_t$ . In fact, it is easy to calculate that

$$(3.8) \quad \begin{aligned} [\tau_t(W(f)), W(g)] &= \{W(T_t f) - W(g)W(T_t f)W(-g)\} W(g) \\ &= \left\{1 - e^{i\sigma(T_t f, g)}\right\} W(T_t f)W(g), \end{aligned}$$

using the Weyl relations (3.4). For the examples we consider below, one can prove that for every  $a > 0$ , there exists positive numbers  $c_a$  and  $v_a$  for which

$$(3.9) \quad |\sigma(T_t f, g)| \leq c_a e^{v_a |t|} \sum_{x, y \in \mathbb{Z}^d} |f(x)| |g(y)| \frac{e^{-a|x-y|}}{(1 + |x-y|)^{d+1}}$$

holds for all  $t \in \mathbb{R}$  and all  $f, g \in \ell^2(\mathbb{Z}^d)$ . In general, we will assume that the harmonic dynamics satisfies an estimate of this type. Namely, we suppose that there exists a number  $a_0 > 0$  for which given  $0 < a \leq a_0$ , there are numbers  $c_a$  and  $v_a$  for which

$$(3.10) \quad \left|1 - e^{i\sigma(T_t f, g)}\right| \leq c_a e^{v_a |t|} \sum_{x, y \in \Gamma} |f(x)| |g(y)| F_a(d(x, y))$$

holds for all  $t \in \mathbb{R}$  and all  $f, g \in \ell^2(\Gamma)$ . Here we describe the spatial decay in  $\Gamma$  through the functions  $F_a$  as introduced in Section 2. Since the Weyl operators are unitary, the norm estimate

$$(3.11) \quad \|[\tau_t(W(f)), W(g)]\| \leq c_a e^{v_a |t|} \sum_{x, y} |f(x)| |g(y)| F_a(d(x, y)),$$

readily follows.

**3.4. An important example.** Using the example given below, we illustrate the general discussion above in terms of a standard harmonic model defined over  $\Gamma = \mathbb{Z}^d$ . We begin with a description of some well known calculations that are valid for these models when restricted to a finite volume. This analysis motivates the definition of the harmonic dynamics in the infinite volume. We then demonstrate that this infinite volume dynamics satisfies a Lieb-Robinson bound. By representing this dynamics in a suitable state, the relevant weak-continuity is readily verified. Interestingly, our analysis also applies to the massless case of  $\omega = 0$ , see below, and we discuss this briefly. We end this subsection with some final comments.

**3.4.1. Finite volume analysis.** We consider a system of coupled harmonic oscillators restricted to a finite volume. Specifically on cubic subsets  $\Lambda_L = (-L, L]^d \subset \mathbb{Z}^d$ , we analyze Hamiltonians of the form

$$(3.12) \quad H_L^h = \sum_{x \in \Lambda_L} p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^d \lambda_j (q_x - q_{x+e_j})^2$$

acting in the Hilbert space

$$(3.13) \quad \mathcal{H}_{\Lambda_L} = \bigotimes_{x \in \Lambda_L} L^2(\mathbb{R}, dq_x).$$

Here the quantities  $p_x$  and  $q_x$ , which appear in (3.12) above, are the single site momentum and position operators regarded as operators on the full Hilbert space  $\mathcal{H}_{\Lambda_L}$  by setting

$$(3.14) \quad p_x = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes -i \frac{d}{dq} \otimes \mathbb{1} \cdots \otimes \mathbb{1} \quad \text{and} \quad q_x = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes q \otimes \mathbb{1} \cdots \otimes \mathbb{1},$$

i.e., these operators act non-trivially only in the  $x$ -th factor of  $\mathcal{H}_{\Lambda_L}$ . These operators satisfy the canonical commutation relations

$$(3.15) \quad [p_x, p_y] = [q_x, q_y] = 0 \quad \text{and} \quad [q_x, p_y] = i\delta_{x,y},$$

valid for all  $x, y \in \Lambda_L$ . In addition,  $\{e_j\}_{j=1}^d$  are the canonical basis vectors in  $\mathbb{Z}^d$ , the numbers  $\lambda_j \geq 0$  and  $\omega \geq 0$  are the parameters of the system, and the Hamiltonian is assumed to have periodic boundary conditions, in the sense that  $q_{x+e_j} = q_{x-(2L-1)e_j}$  if  $x \in \Lambda_L$  but  $x+e_j \notin \Lambda_L$ . It is well-known that Hamiltonians of this form can be diagonalized in Fourier space. We review this quickly to establish some notation and refer the interested reader to [19] for more details.

Introducing the operators

$$(3.16) \quad Q_k = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{x \in \Lambda_L} e^{-ik \cdot x} q_x \quad \text{and} \quad P_k = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{x \in \Lambda_L} e^{-ik \cdot x} p_x,$$

defined for each  $k \in \Lambda_L^* = \left\{ \frac{x\pi}{L} : x \in \Lambda_L \right\}$ , and setting

$$(3.17) \quad \gamma(k) = \sqrt{\omega^2 + 4 \sum_{j=1}^d \lambda_j \sin^2(k_j/2)},$$

one finds that

$$(3.18) \quad H_L^h = \sum_{k \in \Lambda_L^*} \gamma(k) (2b_k^* b_k + 1)$$

where the operators  $b_k$  and  $b_k^*$  satisfy

$$(3.19) \quad b_k = \frac{1}{\sqrt{2\gamma(k)}} P_k - i \sqrt{\frac{\gamma(k)}{2}} Q_k \quad \text{and} \quad b_k^* = \frac{1}{\sqrt{2\gamma(k)}} P_{-k} + i \sqrt{\frac{\gamma(k)}{2}} Q_{-k}.$$

In this sense, we regard the Hamiltonian  $H_L^h$  as diagonalizable.

Using the above diagonalization, one can determine the action of the dynamics corresponding to  $H_L^h$  on the Weyl algebra  $\mathcal{W}(\ell^2(\Lambda_L))$ . In fact, by setting

$$(3.20) \quad W(f) = \exp \left[ i \sum_{x \in \Lambda_L} \operatorname{Re}[f(x)] q_x + \operatorname{Im}[f(x)] p_x \right],$$

for each  $f \in \ell^2(\Lambda_L)$ , it is easy to verify that (3.3) and (3.4) hold with  $\sigma(f, g) = \operatorname{Im}[\langle f, g \rangle]$ . It is convenient to express these Weyl operators in terms of annihilation and creation operators, i.e.,

$$(3.21) \quad a_x = \frac{1}{\sqrt{2}}(q_x + ip_x) \quad \text{and} \quad a_x^* = \frac{1}{\sqrt{2}}(q_x - ip_x),$$

which satisfy

$$(3.22) \quad [a_x, a_y] = [a_x^*, a_y^*] = 0 \quad \text{and} \quad [a_x, a_y^*] = \delta_{x,y} \quad \text{for all } x, y \in \Lambda_L.$$

One finds that

$$(3.23) \quad W(f) = \exp \left[ \frac{i}{\sqrt{2}} (a(f) + a^*(f)) \right],$$

where, for each  $f \in \ell^2(\Lambda_L)$ , we have set

$$(3.24) \quad a(f) = \sum_{x \in \Lambda_L} \overline{f(x)} a_x, \quad a^*(f) = \sum_{x \in \Lambda_L} f(x) a_x^*.$$

Now, the dynamics corresponding to  $H_L^h$ , which we denote by  $\tau_t^L$ , is trivial with respect to the diagonalizing variables, i.e.,

$$(3.25) \quad \tau_t^L(b_k) = e^{-2i\gamma(k)t} b_k \quad \text{and} \quad \tau_t^L(b_k^*) = e^{2i\gamma(k)t} b_k^*,$$

where  $b_k$  and  $b_k^*$  are as defined in (3.19). Hence, if we further introduce

$$(3.26) \quad b_x = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k \in \Lambda_L^*} e^{ikx} b_k \quad \text{and} \quad b_x^* = \frac{1}{\sqrt{|\Lambda_L|}} \sum_{k \in \Lambda_L^*} e^{ikx} b_k^*,$$

for each  $x \in \Lambda_L$  and, analogously to (3.24), define

$$(3.27) \quad b(f) = \sum_{x \in \Lambda_L} \overline{f(x)} b_x, \quad b^*(f) = \sum_{x \in \Lambda_L} f(x) b_x^*,$$

for each  $f \in \ell^2(\Lambda_L)$ , then one has that

$$(3.28) \quad \tau_t^L(b(f)) = b([\mathcal{F}^{-1} M_t \mathcal{F}] f),$$

where  $\mathcal{F}$  is the unitary Fourier transform on  $\ell^2(\Lambda_L)$  and  $M_t$  is the operator of multiplication by  $e^{2i\gamma(k)t}$  in Fourier space with  $\gamma(k)$  as in (3.17). We need only determine the relation between the  $a$ 's and the  $b$ 's.

A short calculation shows that there exists a linear mapping  $U : \ell^2(\Lambda_L) \rightarrow \ell^2(\Lambda_L)$  and an anti-linear mapping  $V : \ell^2(\Lambda_L) \rightarrow \ell^2(\Lambda_L)$  for which

$$(3.29) \quad b(f) = a(Uf) + a^*(Vf),$$

a relation known in the literature as a Bogoliubov transformation [13]. In fact, one has that

$$(3.30) \quad U = \frac{i}{2} \mathcal{F}^{-1} M_{\Gamma_+} \mathcal{F} \quad \text{and} \quad V = \frac{i}{2} \mathcal{F}^{-1} M_{\Gamma_-} \mathcal{F} J$$

where  $J$  is complex conjugation and  $M_{\Gamma_{\pm}}$  is the operator of multiplication by

$$(3.31) \quad \Gamma_{\pm}(k) = \frac{1}{\sqrt{\gamma(k)}} \pm \sqrt{\gamma(k)},$$

with  $\gamma(k)$  as in (3.17). Using the fact that  $\Gamma_\pm$  is real valued and even, it is easy to check that

$$(3.32) \quad U^*U - V^*V = \mathbb{1} = UU^* - VV^*$$

and

$$(3.33) \quad V^*U - U^*V = 0 = VU^* - UV^*$$

where we stress that  $V^*$  is the adjoint of the *anti-linear* mapping  $V$ . The relation (3.29) is invertible, in fact,

$$(3.34) \quad a(f) = b(U^*f) - b^*(V^*f),$$

and therefore

$$(3.35) \quad W(f) = \exp \left[ \frac{i}{\sqrt{2}} (b((U^* - V^*)f) + b^*((U^* - V^*)f)) \right].$$

Clearly then,

$$(3.36) \quad \tau_t(W(f)) = W(T_t f),$$

where the mapping  $T_t$  is given by

$$(3.37) \quad T_t = (U + V)\mathcal{F}^{-1}M_t\mathcal{F}(U^* - V^*),$$

and we have used (3.28).

**3.4.2. Infinite volume dynamics.** It is now clear how to define the infinite volume harmonic dynamics. Consider a subspace  $\mathcal{D} \subset \ell^2(\mathbb{Z}^d)$  and define  $\mathcal{W}(\mathcal{D})$  as above with  $\sigma(f, g) = \text{Im}[\langle f, g \rangle]$ . First assume  $\omega > 0$ , take  $\gamma : [-\pi, \pi]^d \rightarrow \mathbb{R}$  as in (3.17), and set  $U$  and  $V$  as in (3.30) with (3.31). If  $\omega > 0$ , both  $U$  and  $V$  are bounded transformations on  $\ell^2(\mathbb{Z}^d)$ . We will treat the case  $\omega = 0$  by a limiting argument. The mapping  $T_t$  defined by setting

$$(3.38) \quad T_t = (U + V)\mathcal{F}^{-1}M_t\mathcal{F}(U^* - V^*),$$

is well-defined on  $\ell^2(\mathbb{Z}^d)$ . To define the dynamics on  $\mathcal{W}(\mathcal{D})$  we will need to choose subspaces  $\mathcal{D}$  that are  $T_t$  invariant. On such  $\mathcal{D}$ ,  $T_t$  is clearly real-linear. With (3.32) and (3.33), one can easily verify the group properties  $T_0 = \mathbb{1}$ ,  $T_{s+t} = T_s \circ T_t$ , and

$$(3.39) \quad \text{Im}[\langle T_t f, T_t g \rangle] = \text{Im}[\langle f, g \rangle],$$

i.e.  $T_t$  is symplectic in the sense of (3.6). Using Theorem 5.2.8 of [4], there is a unique one parameter group of  $*$ -automorphisms on  $\mathcal{W}(\mathcal{D})$ , which we will denote by  $\tau_t$ , that satisfies

$$(3.40) \quad \tau_t(W(f)) = W(T_t f) \quad \text{for all } f \in \mathcal{D}.$$

This defines the harmonic dynamics on  $\mathcal{W}(\mathcal{D})$ .

Here it is important that  $T_t : \mathcal{D} \rightarrow \mathcal{D}$ . As we demonstrated in [19], the mapping  $T_t$  can be expressed as a convolution. In fact,

$$(3.41) \quad T_t f = f * \overline{\left( H_t^{(0)} + \frac{i}{2}(H_t^{(-1)} + H_t^{(1)}) \right)} + \bar{f} * \left( \frac{i}{2}(H_t^{(1)} - H_t^{(-1)}) \right).$$

where

$$(3.42) \quad \begin{aligned} H_t^{(-1)}(x) &= \frac{1}{(2\pi)^d} \text{Im} \left[ \int \frac{1}{\gamma(k)} e^{i(k \cdot x - 2\gamma(k)t)} dk \right], \\ H_t^{(0)}(x) &= \frac{1}{(2\pi)^d} \text{Re} \left[ \int e^{i(k \cdot x - 2\gamma(k)t)} dk \right], \\ H_t^{(1)}(x) &= \frac{1}{(2\pi)^d} \text{Im} \left[ \int \gamma(k) e^{i(k \cdot x - 2\gamma(k)t)} dk \right]. \end{aligned}$$

Using analysis similar to what is proven in [19], the following result holds.

**Lemma 3.1.** *Consider the functions defined in (3.42). For  $\omega \geq 0, \lambda_1, \dots, \lambda_d \geq 0$ , but such that  $c_{\omega, \lambda} = (\omega^2 + 4 \sum_{j=1}^d \lambda_j)^{1/2} > 0$ , and any  $\mu > 0$ , the bounds*

$$(3.43) \quad \begin{aligned} |H_t^{(0)}(x)| &\leq e^{-\mu(|x| - c_{\omega, \lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)} \\ |H_t^{(-1)}(x)| &\leq c_{\omega, \lambda}^{-1} e^{-\mu(|x| - c_{\omega, \lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)} \\ |H_t^{(1)}(x)| &\leq c_{\omega, \lambda} e^{\mu/2} e^{-\mu(|x| - c_{\omega, \lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)} \end{aligned}$$

hold for all  $t \in \mathbb{R}$  and  $x \in \mathbb{Z}^d$ . Here  $|x| = \sum_{j=1}^d |x_j|$ .

Given the estimates in Lemma 3.1, equation (3.41) and Young's inequality imply that  $T_t$  can be defined as a transformation of  $\ell^p(\mathbb{Z}^d)$ , for  $p \geq 1$ . However, the symplectic form limits us to consider  $\mathcal{D} = \ell^p(\mathbb{Z}^d)$  with  $1 \leq p \leq 2$ .

The following bound now readily follows:

$$(3.44) \quad \begin{aligned} |\operatorname{Im}\langle T_t f, g \rangle| &\leq \left(1 + 2e^{\mu/2} c_{\omega, \lambda} + 2c_{\omega, \lambda}^{-1}\right) \times \\ &\times \sum_{x, y} |f(x)| |g(y)| e^{-\mu(|x| - c_{\omega, \lambda} \max(\frac{2}{\mu}, e^{(\mu/2)+1})|t|)} \end{aligned}$$

This implies an estimate of the form (3.9), and hence a Lieb-Robinson bound as in (3.11).

A simple corollary of Lemma 3.1 follows.

**Corollary 3.2.** *Consider the functions defined in (3.42). For  $\omega \geq 0, \lambda_1, \dots, \lambda_d \geq 0$ , but with  $c_{\omega, \lambda} = (\omega^2 + 4 \sum_{j=1}^d \lambda_j)^{1/2} > 0$ . Take  $\|\cdot\|_1$  to be the  $\ell^1$ -norm. One has that*

$$(3.45) \quad \|H_t^{(0)} - \delta_0\|_1 \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

and

$$(3.46) \quad \|H_t^{(m)}\|_1 \rightarrow 0 \quad \text{as } t \rightarrow 0, \quad \text{for } m \in \{-1, 1\}.$$

*Proof.* The estimates in Lemma 3.1 imply that the functions  $H_t^{(m)}$  are bounded by exponentially decaying functions (in  $|x|$ ). These estimates are uniform for  $t$  in compact sets, e.g.  $t \in [-1, 1]$ , and therefore dominated convergence applies. It is clear that  $H_0^{(0)}(x) = \delta_0(x)$  while  $H_0^{(m)}(x) = 0$  for  $m \in \{-1, 1\}$ . This proves the corollary.  $\square$

**3.4.3. Representing the dynamics.** The infinite-volume ground state of the model (3.12) is the vacuum state for the  $b$ -operators, as can be seen from (3.18). This state is defined on  $\mathcal{W}(\mathcal{D})$  by

$$(3.47) \quad \rho(W(f)) = e^{-\frac{1}{4}\|(U^* - V^*)f\|^2}$$

By standard arguments this defines a state on  $\mathcal{W}(\mathcal{D})$  [4]. Using (3.38), (3.32) and (3.33) one readily verifies that  $\rho$  is  $\tau_t$ -invariant.  $\rho$  is regular by observation. The weak continuity of the dynamics in the GNS-representation of  $\rho$  will follow from the continuity of the functions of the form

$$(3.48) \quad t \mapsto \rho(W(g_1)W(T_t f)W(g_2)), \quad \text{for } g_1, g_2, f \in \mathcal{D}.$$

When  $\omega > 0$ , this continuity can be easily observed from the following expression:

$$(3.49) \quad \begin{aligned} \rho(W(g_1)W(T_t f)W(g_2)) &= e^{i\sigma(g_1, g_2)/2} e^{i\sigma(T_t f, g_2 - g_1)/2} \times \\ &\times e^{-\|(U^* - V^*)(g_1 + g_2 + T_t f)\|^2/4} \end{aligned}$$

Note that  $T_t$  is differentiable with bounded derivative and that both  $U$  and  $V$  are bounded. This establishes the continuity in the case that  $\omega > 0$ .

As discussed in the introduction of the section, the  $W^*$ -dynamical system is now defined by considering the GNS representation  $\pi_\rho$  of  $\rho$ . This yields a von Neumann algebra  $\mathcal{M} = \overline{\pi_\rho(\mathcal{W}(\mathcal{D}))}$ . The invariance of  $\rho$  implies that the dynamics is implementable by unitaries  $U_t$ , i.e.,

$$(3.50) \quad \pi_\rho(\tau_t(W(f))) = U_t^* \pi_\rho(W(f)) U_t.$$

Using  $U_t$ , the dynamics can be extended to  $\mathcal{M}$ . As a consequence of (3.48), this extended dynamics is weakly continuous.

**3.4.4. The case of  $\omega = 0$ .** We now discuss the case  $\omega = 0$ . Here, the maps  $T_t$  are defined using the convolution formula (3.41). By Lemma 3.1,  $T_t$  is well-defined as a transformation of  $\ell^p(\mathbb{Z}^d)$ , for  $1 \leq p \leq 2$ . Both the group property of  $T_t$  and the invariance of the symplectic form  $\sigma$  follow in the limit  $\omega \rightarrow 0$  by dominated convergence which is justified by Lemma 3.1. This demonstrates that the dynamics is well defined.

We represent the dynamics in a state  $\rho$  is defined by (3.47), but with the understanding that  $\|(U^* - V^*)f\|$  may take on the value  $+\infty$ , in which case  $\rho(W(f)) = 0$ .  $\rho$  is still clearly regular. It remains to show that the dynamics is weakly continuous.

Observe that

$$(3.51) \quad \begin{aligned} T_t f - f &= f * \left( H_t^{(0)} - \delta_0 \right) - f * \left( \frac{i}{2} (H_t^{(-1)} + H_t^{(1)}) \right) \\ &\quad + \bar{f} * \left( \frac{i}{2} (H_t^{(1)} - H_t^{(-1)}) \right), \end{aligned}$$

follows from (3.41). Using Young's inequality and Corollary 3.2, it is clear that  $\|T_t f - f\| \rightarrow 0$  as  $t \rightarrow 0$  for any  $f \in \ell^p(\mathbb{Z}^d)$  with  $1 \leq p \leq 2$ . A calculation shows that

$$(3.52) \quad (U^* - V^*)(T_t f - f) = F_1 * \left( H_t^{(0)} - \delta_0 \right) - F_2 * H_t^{(-1)} - i F_3 * H_t^{(1)},$$

where

$$(3.53) \quad \begin{aligned} F_1 &= \mathcal{F}^{-1} M_{\sqrt{\gamma}} \mathcal{F} \operatorname{Im}[f] - i \mathcal{F}^{-1} M_{\gamma^{-1/2}} \mathcal{F} \operatorname{Re}[f], \\ F_2 &= \mathcal{F}^{-1} M_{\sqrt{\gamma}} \mathcal{F} \operatorname{Re}[f], \quad \text{and} \quad F_3 = \mathcal{F}^{-1} M_{\gamma^{-1/2}} \mathcal{F} \operatorname{Im}[f]. \end{aligned}$$

A similar argument to what is given above now implies that  $\|(U^* - V^*)(T_t f - f)\| \rightarrow 0$  as  $t \rightarrow 0$ , for any  $f \in \mathcal{D}_0$ , where

$$(3.54) \quad \mathcal{D}_0 = \left\{ f \in \ell^2(\mathbb{Z}^d) : \mathcal{F}^{-1} M_{\gamma^{-1/2}} \mathcal{F} \operatorname{Re}[f] \in \ell^2(\mathbb{Z}^d) \right\}.$$

No additional assumption on  $\operatorname{Im}[f]$  is necessary since  $F_3$  is convolved with  $H_t^{(1)}$ . Given the form of (3.49), this suffices to prove weak continuity. In fact, one can check that  $T_t$  leaves  $\mathcal{D}_0$  invariant and that if  $f \in \mathcal{D}_0$ , then  $(U^* - V^*)T_t f \in \ell^2(\mathbb{Z}^d)$  for all  $t \in \mathbb{R}$ . This establishes weak continuity of the dynamics, defined on  $\mathcal{W}(\mathcal{D}_0)$ .

*Remark 3.3.* We observe that, when  $\omega = 0$ , the finite volume Hamiltonian  $H_L^h$  (3.12) is translation invariant and commutes with the total momentum operator  $P_0$  (see (3.16)). In fact,  $H_L^h$  can be written as

$$\begin{aligned} H_L^h &= P_0^2 + \sum_{k \in \Lambda_L^* \setminus \{0\}} P_k^* P_k + \gamma^2(k) Q_k^* Q_k \\ &= P_0^2 + \sum_{k \in \Lambda_L^* \setminus \{0\}} \gamma(k) (2b_k^* b_k + 1) \end{aligned}$$

where we used the notation (3.16) and, for  $k \neq 0$ , we introduced the operators  $b_k, b_k^*$  as in (3.19). In this case, the operator  $H_L^h$  does not have eigenvectors; its spectrum is purely continuous. By a unitary transformation, the Hilbert space  $\mathcal{H}_{\Lambda_L}$  (see (3.13)) can be mapped into the space  $L^2(\mathbb{R}, dP_0; \mathcal{H}_b)$

of square integrable functions of  $P_0 \in \mathbb{R}$ , with values in  $\mathcal{H}_b$ . Here,  $\mathcal{H}_b$  denotes the Fock space generated by all creation and annihilation operators  $b_k^*, b_k$  with  $k \neq 0$ . It is then easy to construct vectors which minimize the energy by a given distribution of the total momentum; for an arbitrary (complex valued)  $f \in L^2(\mathbb{R})$  with  $\|f\| = 1$ , we define  $\psi_f \in L^2(\mathbb{R}, dP_0; \mathcal{H}_b)$  by setting  $\psi_f(P_0) = f(P_0)\Omega$  (where  $\Omega$  is the Fock vacuum in  $\mathcal{H}_b$ ). These vectors are not invariant with respect to the time evolution. It is simple to check that the Schrödinger evolution of  $\psi_f$  is given by  $e^{-iH_L^h t} \psi_f = \psi_{f_t}$  with  $f_t(P_0) = e^{-itP_0^2} f(P_0)$  is the free evolution of  $f$ . In particular, for  $\omega = 0$ ,  $H_L^h$  does not have a ground state in the traditional sense of an eigenvector. For this reason, when  $\omega = 0$ , it is not a priori clear what the natural choice of state should be. As is discussed above, one possibility is to consider first  $\omega \neq 0$  and then take the limit  $\omega \rightarrow 0$ . This yields a ground state for the infinite system with vanishing center of mass momentum of the oscillators. By considering non-zero values for the center of mass momentum, one can also define other states with similar properties.

**3.4.5. Some final comments.** The analysis in the following sections and our main result is not limited to the class of examples we discussed above. E.g., harmonic systems defined on more general graphs, such as the ones considered in [6, 7] can also be treated. Also note that our choice of time-invariant state, while natural, is by no means the only possible. Instead of the vacuum state defined in (3.47), equilibrium states at positive temperatures could be used in exactly the same way. It would also make sense to study the convergence of the equilibrium or ground states for the perturbed dynamics and to consider the dynamics in the representation of the limiting infinite-system state, but we have not studied this situation and will not discuss it in this paper.

#### 4. PERTURBING THE HARMONIC DYNAMICS

In this section, we will discuss finite volume perturbations of the infinite volume harmonic dynamics which we defined in Section 3. To begin, we recall a fundamental result about perturbations of quantum dynamics defined by adding a bounded term to the generator. This is a version of what is usually known as the Dyson or Duhamel expansion. The following statement summarizes Proposition 5.4.1 of [4].

**Proposition 4.1.** *Let  $\{\mathcal{M}, \alpha_t\}$  be a  $W^*$ -dynamical system and let  $\delta$  denote the infinitesimal generator of  $\alpha_t$ . Given any  $P = P^* \in \mathcal{M}$ , set  $\delta_P$  to be the bounded derivation with domain  $D(\delta_P) = \mathcal{M}$  satisfying  $\delta_P(A) = i[P, A]$  for all  $A \in \mathcal{M}$ . It follows that  $\delta + \delta_P$  generates a one-parameter group of  $*$ -automorphisms  $\alpha^P$  of  $\mathcal{M}$  which is the unique solution of the integral equation*

$$(4.1) \quad \alpha_t^P(A) = \alpha_t(A) + i \int_0^t \alpha_s^P([P, \alpha_{t-s}(A)]) ds.$$

*In addition, the estimate*

$$(4.2) \quad \|\alpha_t^P(A) - \alpha_t(A)\| \leq \left(e^{|t|\|P\|} - 1\right) \|A\|$$

*holds for all  $t \in \mathbb{R}$  and  $A \in \mathcal{M}$ .*

Since the initial dynamics  $\alpha_t$  is assumed weakly continuous, the norm estimate (4.2) can be used to show that the perturbed dynamics is also weakly continuous. Hence, for each  $P = P^* \in \mathcal{M}$  the pair  $\{\mathcal{M}, \alpha_t^P\}$  is also a  $W^*$ -dynamical system. Thus, if  $P_i = P_i^* \in \mathcal{M}$  for  $i = 1, 2$ , then one can define  $\alpha_t^{P_1+P_2}$  iteratively.

**4.1. A Lieb-Robinson bound for on-site perturbations.** In this section we will consider perturbations of the harmonic dynamics defined in Section 3. Recall that our general assumptions for the harmonic dynamics on  $\Gamma$  are as follows.

We assume that the harmonic dynamics,  $\tau_t^0$ , is defined on a Weyl algebra  $\mathcal{W}(\mathcal{D})$  where  $\mathcal{D}$  is a subspace of  $\ell^2(\Gamma)$ . In fact, we assume there exists a group  $T_t$  of real-linear transformations which leave  $\mathcal{D}$  invariant and satisfy

$$(4.3) \quad \tau_t^0(W(f)) = W(T_t f) \quad \text{for all } f \in \mathcal{D}.$$

In addition, we assume that this harmonic dynamics satisfies a Lieb-Robinson bound. Specifically, we suppose that there exists a number  $a_0 > 0$  for which given any  $0 < a \leq a_0$ , there are positive numbers  $c_a$  and  $v_a$  for which

$$(4.4) \quad \left| 1 - e^{i\sigma(T_t f, g)} \right| \leq c_a e^{v_a |t|} \sum_{x, y \in \Gamma} |f(x)| |g(y)| F_a(d(x, y))$$

here the spatial decay in  $\Gamma$  is described by the function  $F_a$  as introduced in Section 2. As we discussed in Section 3, the estimate (4.4) immediately implies the Lieb-Robinson bound

$$(4.5) \quad \left\| [\tau_t^0(W(f)), W(g)] \right\| \leq c_a e^{v_a |t|} \sum_{x, y \in \Gamma} |f(x)| |g(y)| F_a(d(x, y)).$$

Finally, we assume that we have represented this harmonic dynamics in a regular and  $\tau_t^0$ -invariant state  $\rho$  for which the pair  $\{\mathcal{M}, \tau_t^0\}$ , with  $\mathcal{M} = \overline{\pi_\rho(\mathcal{W}(\mathcal{D}))}$ , is a  $W^*$ -dynamical system.

Our first estimate involves perturbations defined as finite sums of on-site terms. More specifically, the perturbations we consider are defined as follows.

To each site  $x \in \Gamma$ , we will associate a finite measure  $\mu_x$  on  $\mathbb{C}$ , and an element  $P_x \in \mathcal{W}(\mathcal{D})$  which has the form

$$(4.6) \quad P_x = \int_{\mathbb{C}} W(z\delta_x) \mu_x(dz).$$

We require that each  $\mu_x$  is even, i.e. invariant under  $z \mapsto -z$ , to ensure self-adjointness, i.e.  $P_x^* = P_x$ . Our Lieb-Robinson bounds hold under the additional assumption that the second moment is uniformly bounded, i.e.

$$(4.7) \quad \sup_{x \in \Gamma} \int_{\mathbb{C}} |z|^2 |\mu_x|(dz) < \infty.$$

We use Proposition 4.1 to define the perturbed dynamics. Fix a finite set  $\Lambda \subset \Gamma$ . Set

$$(4.8) \quad P^\Lambda = \sum_{x \in \Lambda} P_x,$$

and note that  $(P^\Lambda)^* = P^\Lambda \in \mathcal{W}(\mathcal{D})$ . We will denote by  $\tau_t^{(\Lambda)}$  the dynamics that results from applying Proposition 4.1 to the  $W^*$ -dynamical system  $\{\mathcal{M}, \tau_t^0\}$  and  $P^\Lambda$ .

Before we begin the proof of our estimate, we discuss two examples.

*Example. 1)* Let  $\mu_x$  be supported on  $[-\pi, \pi)$  and absolutely continuous with respect to Lebesgue measure, i.e.  $\mu_x(dz) = v_x(z)dz$ . If  $v_x$  is in  $L^2([-\pi, \pi))$ , then  $P_x$  is proportional to an operator of multiplication by the inverse Fourier transform of  $v_x$ . Moreover, since the support of  $\mu_x$  is real,  $P_x$  corresponds to multiplication by a function depending only on  $q_x$ .

*Example. 2)* Let  $\mu_x$  have finite support, e.g., take  $\text{supp}(\mu_x) = \{z, -z\}$  for some number  $z = \alpha + i\beta \in \mathbb{C}$ . Then

$$(4.9) \quad P_x = W(z\delta_x) + W(-z\delta_x) = 2 \cos(\alpha q_x + \beta p_x).$$

We now state our first result.

**Theorem 4.2.** *Let  $\tau_t^0$  be a harmonic dynamics defined on  $\Gamma$  as described above. Suppose that*

$$(4.10) \quad \kappa = \sup_{x \in \Gamma} \int_{\mathbb{C}} |z|^2 |\mu_x|(dz) < \infty,$$

*and define the perturbed dynamics  $\tau_t^{(\Lambda)}$  as indicated above. For every  $0 < a \leq a_0$ , there exist positive numbers  $c_a$  and  $v_a$  for which the estimate*

$$(4.11) \quad \left\| \left[ \tau_t^{(\Lambda)}(W(f)), W(g) \right] \right\| \leq c_a e^{(v_a + c_a \kappa C_a)|t|} \sum_{x,y} |f(x)| |g(y)| F_a(d(x, y))$$

*holds for all  $t \in \mathbb{R}$  and for any functions  $f, g \in \mathcal{D}$ .*

Here the numbers  $c_a$  and  $v_a$  are as in (4.4), whereas  $C_a$  is the convolution constant as defined in (2.2) with respect to the function  $F_a$ .

*Proof.* Fix  $t > 0$  and define the function  $\Psi_t : [0, t] \rightarrow \mathcal{W}(\mathcal{D})$  by setting

$$(4.12) \quad \Psi_t(s) = \left[ \tau_s^{(\Lambda)}(\tau_{t-s}^0(W(f))), W(g) \right].$$

It is clear that  $\Psi_t$  interpolates between the commutator associated with the original harmonic dynamics,  $\tau_t^0$  at  $s = 0$ , and that of the perturbed dynamics,  $\tau_t^{(\Lambda)}$  at  $s = t$ . A calculation shows that

$$(4.13) \quad \frac{d}{ds} \Psi_t(s) = i \sum_{x \in \Lambda} \left[ \tau_s^{(\Lambda)}([P_x, W(T_{t-s}f)]), W(g) \right],$$

where differentiability is guaranteed by the results of Proposition 4.1. The inner commutator can be expressed as

$$(4.14) \quad \begin{aligned} [P_x, W(T_{t-s}f)] &= \int_{\mathbb{C}} [W(z\delta_x), W(T_{t-s}f)] \mu_x(dz) \\ &= W(T_{t-s}f) \mathcal{L}_{t-s;x}(f). \end{aligned}$$

where

$$(4.15) \quad \mathcal{L}_{t-s;x}^*(f) = \mathcal{L}_{t-s;x}(f) = \int_{\mathbb{C}} W(z\delta_x) \left\{ e^{i\sigma(T_{t-s}f, z\delta_x)} - 1 \right\} \mu_x(dz) \in \mathcal{W}(\mathcal{D}).$$

Thus  $\Psi_t$  satisfies

$$(4.16) \quad \begin{aligned} \frac{d}{ds} \Psi_t(s) &= i \sum_{x \in \Lambda} \Psi_t(s) \tau_s^{(\Lambda)}(\mathcal{L}_{t-s;x}(f)) \\ &\quad + i \sum_{x \in \Lambda} \tau_s^{(\Lambda)}(W(T_{t-s}f)) \left[ \tau_s^{(\Lambda)}(\mathcal{L}_{t-s;x}(f)), W(g) \right]. \end{aligned}$$

The first term above is norm preserving. In fact, define a unitary evolution  $U_t(\cdot)$  by setting

$$(4.17) \quad \frac{d}{ds} U_t(s) = -i \sum_{x \in \Lambda} \tau_s^{(\Lambda)}(\mathcal{L}_{t-s;x}(f)) U_t(s) \quad \text{with } U_t(0) = \mathbb{1}.$$

It is easy to see that

$$(4.18) \quad \frac{d}{ds} (\Psi_t(s) U_t(s)) = i \sum_{x \in \Lambda} \tau_s^{(\Lambda)}(W(T_{t-s}f)) \left[ \tau_s^{(\Lambda)}(\mathcal{L}_{t-s;x}(f)), W(g) \right] U_t(s),$$

and therefore,

$$(4.19) \quad \Psi_t(t) U_t(t) = \Psi_t(0) + i \sum_{x \in \Lambda} \int_0^t \tau_s^{(\Lambda)}(W(T_{t-s}f)) \left[ \tau_s^{(\Lambda)}(\mathcal{L}_{t-s;x}(f)), W(g) \right] U_t(s) ds.$$

Estimating in norm, we find that

$$(4.20) \quad \left\| \left[ \tau_t^{(\Lambda)} (W(f)), W(g) \right] \right\| \leq \left\| \left[ \tau_t^0 (W(f)), W(g) \right] \right\| + \sum_{x \in \Lambda} \int_0^t \left\| \left[ \tau_s^{(\Lambda)} (\mathcal{L}_{t-s;x}(f)), W(g) \right] \right\| ds.$$

Moreover, using (4.15) and the bound (4.4), it is clear that

$$(4.21) \quad \left\| \left[ \tau_s^{(\Lambda)} (\mathcal{L}_{t-s;x}(f)), W(g) \right] \right\| \leq c_a e^{v_a(t-s)} \sum_{x' \in \Gamma} |f(x')| F_a(d(x, x')) \times \int_{\mathbb{C}} |z| \left\| \left[ \tau_s^{(\Lambda)} (W(z\delta_x)), W(g) \right] \right\| |\mu_x|(dz)$$

holds. Combining (4.21), (4.20), and (4.5), we have proven that

$$(4.22) \quad \left\| \left[ \tau_t^{(\Lambda)} (W(f)), W(g) \right] \right\| \leq c_a e^{v_a t} \sum_{x,y} |f(x)| |g(y)| F_a(d(x, y)) + c_a \sum_{x' \in \Gamma} |f(x')| \sum_{x \in \Lambda} F_a(d(x, x')) \int_0^t e^{v_a(t-s)} \times \int_{\mathbb{C}} |z| \left\| \left[ \tau_s^{(\Lambda)} (W(z\delta_x)), W(g) \right] \right\| |\mu_x|(dz) ds.$$

Following the iteration scheme applied in [19], one arrives at (4.11) as claimed.  $\square$

**4.2. Multiple Site Anharmonicities.** In this section, we will prove that Lieb-Robinson bounds, similar to those in Theorem 4.2, also hold for perturbations involving short range interactions. We introduce these as follows.

For each finite subset  $X \subset \Gamma$ , we associate a finite measure  $\mu_X$  on  $\mathbb{C}^X$  and an element  $P_X \in \mathcal{W}(\mathcal{D})$  with the form

$$(4.23) \quad P_X = \int_{\mathbb{C}^X} W(z \cdot \delta_X) \mu_X(dz),$$

where, for each  $z \in \mathbb{C}^X$ , the function  $z \cdot \delta_X : \Gamma \rightarrow \mathbb{C}$  is given by

$$(4.24) \quad (z \cdot \delta_X)(x) = \sum_{x' \in X} z_{x'} \delta_{x'}(x) = \begin{cases} z_x & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We will again require that  $\mu_X$  is invariant with respect to  $z \mapsto -z$ , and hence,  $P_X$  is self-adjoint. In analogy to (4.8), for any finite subset  $\Lambda \subset \Gamma$ , we will set

$$(4.25) \quad P^\Lambda = \sum_{X \subset \Lambda} P_X,$$

where the sum is over all subsets of  $\Lambda$ . Here we will again let  $\tau_t^{(\Lambda)}$  denote the dynamics resulting from Proposition 4.1 applied to the  $W^*$ -dynamical system  $\{\mathcal{M}, \tau_t^0\}$  and the perturbation  $P^\Lambda$  defined by (4.25).

The main assumption on these multi-site perturbations follows. There exists a number  $a_1 > 0$  such that for all  $0 < a \leq a_1$ , there is a number  $\kappa_a > 0$  for which given any pair  $x_1, x_2 \in \Gamma$ ,

$$(4.26) \quad \sum_{\substack{X \subset \Gamma: \\ x_1, x_2 \in X}} \int_{\mathbb{C}^X} |z_{x_1}| |z_{x_2}| |\mu_X|(dz) \leq \kappa_a F_a(d(x_1, x_2)).$$

**Theorem 4.3.** *Let  $\tau_t^0$  be a harmonic dynamics defined on  $\Gamma$ . Assume that (4.26) holds, and that  $\tau_t^{(\Lambda)}$  denotes the corresponding perturbed dynamics. For every  $0 < a \leq \min(a_0, a_1)$ , there exist positive numbers  $c_a$  and  $v_a$  for which the estimate*

$$(4.27) \quad \left\| \left[ \tau_t^{(\Lambda)}(W(f)), W(g) \right] \right\| \leq c_a e^{(v_a + c_a \kappa_a C_a^2)|t|} \sum_{x,y} |f(x)| |g(y)| F_a(d(x, y))$$

holds for all  $t \in \mathbb{R}$  and for any functions  $f, g \in \mathcal{D}$ .

The proof of this result closely follows that of Theorem 4.2, and so we only comment on the differences.

*Proof.* For  $f, g \in \mathcal{D}$  and  $t > 0$ , define  $\Psi_t : [0, t] \rightarrow \mathcal{W}(\mathcal{D})$  as in (4.12). The derivative calculation beginning with (4.13) proceeds as before. Here

$$(4.28) \quad \mathcal{L}_{t-s;X}(f) = \int_{\mathbb{C}^X} W(z \cdot \delta_X) \left\{ e^{i\sigma(T_{t-s}f, z \cdot \delta_X)} - 1 \right\} \mu_X(dz),$$

is also self-adjoint. The norm estimate

$$(4.29) \quad \left\| \left[ \tau_t^{(\Lambda)}(W(f)), W(g) \right] \right\| \leq \left\| \left[ \tau_t^0(W(f)), W(g) \right] \right\| + \sum_{X \subset \Lambda} \int_0^t \left\| \left[ \tau_s^{(\Lambda)}(\mathcal{L}_{t-s;X}(f)), W(g) \right] \right\| ds,$$

holds similarly. With (4.28), it is easy to see that the integrand in (4.29) is bounded by

$$(4.30) \quad c_a e^{v_a(t-s)} \sum_{x \in \Gamma} |f(x)| \sum_{x' \in X} F_a(d(x, x')) \int_{\mathbb{C}^X} |z_{x'}| \left\| \left[ \tau_s^{(\Lambda)}(W(z \cdot \delta_X)), W(g) \right] \right\| |\mu_X|(dz),$$

the analogue of (4.21), for  $0 < a \leq a_0$ . Moreover, if  $0 < a \leq \min(a_0, a_1)$ , then

$$(4.31) \quad \begin{aligned} \left\| \left[ \tau_t^{(\Lambda)}(W(f)), W(g) \right] \right\| &\leq c_a e^{v_a t} \sum_{x,y \in \Gamma} |f(x)| |g(y)| F_a(d(x, y)) \\ &+ c_a \sum_{x \in \Gamma} |f(x)| \sum_{X \subset \Lambda} \sum_{x' \in X} F_a(d(x, x')) \times \\ &\times \int_0^t e^{v_a(t-s)} \int_{\mathbb{C}^X} |z_{x'}| \left\| \left[ \tau_s^{(\Lambda)}(W(z \cdot \delta_X)), W(g) \right] \right\| |\mu_X|(dz) ds. \end{aligned}$$

The estimate claimed in (4.27) follows by iteration. In fact, the first term in the iteration is bounded by

$$\begin{aligned}
(4.32) \quad & c_a \sum_x |f(x)| \sum_{X \subset \Lambda} \sum_{x_1 \in X} F_a(d(x, x_1)) \int_0^t e^{v_a(t-s)} \\
& \quad \times \int_{\mathbb{C}^X} |z_{x_1}| \left( c_a e^{v_a s} \sum_{x_2 \in X} \sum_y |z_{x_2}| |g(y)| F_a(d(x_2, y)) \right) |\mu_X|(dz) ds \\
& \leq c_a t \cdot c_a e^{v_a t} \sum_{x, y} |f(x)| |g(y)| \sum_{x_1, x_2 \in \Gamma} F_a(d(x, x_1)) F_a(d(x_2, y)) \sum_{\substack{X \subset \Gamma: \\ x_1, x_2 \in X}} \int_{\mathbb{C}^X} |z_{x_1}| |z_{x_2}| |\mu_X|(dz) \\
& \leq \kappa_a c_a t \cdot c_a e^{v_a t} \sum_{x, y} |f(x)| |g(y)| \sum_{x_1, x_2 \in \Gamma} F_a(d(x, x_1)) F_a(d(x_1, x_2)) F_a(d(x_2, y)) \\
& \leq \kappa_a C_a^2 c_a t \cdot c_a e^{v_a t} \sum_{x, y} |f(x)| |g(y)| F_a(d(x, y)) .
\end{aligned}$$

The higher order iterates are treated similarly.  $\square$

## 5. EXISTENCE OF THE DYNAMICS

In this section, we demonstrate that the finite volume dynamics analyzed in the previous section converge to a limiting dynamics as the volume  $\Lambda$  on which the perturbation is defined tends to  $\Gamma$ . We state this as Theorem 5.1 below.

**Theorem 5.1.** *Let  $\tau_t^0$  be a harmonic dynamics defined on  $\mathcal{W}(\ell^1(\Gamma))$  as described in Section 4.1. Let  $\{\Lambda_n\}$  denote a non-decreasing, exhaustive sequence of finite subsets of  $\Gamma$ . Consider a family of perturbations  $P^{\Lambda_n}$  as defined in (4.25) and (4.23) which satisfy (4.26). Suppose in addition that*

$$(5.1) \quad M = \sup_{\substack{x \in \Gamma \\ \substack{X \subset \Gamma: \\ x \in X}}} \sum \int_{\mathbb{C}^X} |z_x| |\mu_X|(dz) < \infty .$$

*Then, for each  $f \in \ell^1(\Gamma)$  and  $t \in \mathbb{R}$  fixed, the limit*

$$(5.2) \quad \lim_{n \rightarrow \infty} \tau_t^{(\Lambda_n)}(W(f))$$

*exists in norm. The limiting dynamics, which we denote by  $\tau_t$ , is weakly continuous.*

It is important to note that since the estimates in Theorem 4.3 are independent of  $\Lambda$ , the limiting dynamics also satisfies a Lieb-Robinson bound as in (4.27). We now prove Theorem 5.1.

*Proof.* Fix a Weyl operator  $W(f)$  with  $f \in \ell^1(\Gamma)$ . Let  $T > 0$  and take  $m \leq n$ . Iteratively applying Proposition 4.1, we have that

$$(5.3) \quad \tau_t^{(\Lambda_n)}(W(f)) = \tau_t^{(\Lambda_m)}(W(f)) + i \int_0^t \tau_s^{(\Lambda_n)} \left( \left[ P^{\Lambda_n \setminus \Lambda_m}, \tau_{t-s}^{(\Lambda_m)}(W(f)) \right] \right) ds ,$$

for all  $-T \leq t \leq T$ . The bound

$$\begin{aligned}
(5.4) \quad & \left\| \left[ P^{\Lambda_n \setminus \Lambda_m}, \tau_{t-s}^{(\Lambda_m)}(W(f)) \right] \right\| \\
& \leq \sum_{\substack{X \subset \Lambda_n: \\ X \cap \Lambda_n \setminus \Lambda_m \neq \emptyset}} \int_{\mathbb{C}^X} \left\| \left[ W(z \cdot \delta_X), \tau_{t-s}^{(\Lambda_m)}(W(f)) \right] \right\| |\mu_X|(dz) \\
& \leq c_a e^{(v_a + c_a \kappa_a C_a^2)(t-s)} \sum_{x \in \Gamma} |f(x)| \sum_{\substack{X \subset \Lambda_n: \\ X \cap \Lambda_n \setminus \Lambda_m \neq \emptyset}} \sum_{y \in X} F_a(d(x, y)) \int_{\mathbb{C}^X} |z_y| |\mu_X|(dz) \\
& \leq c_a e^{(v_a + c_a \kappa_a C_a^2)(t-s)} \sum_{x \in \Gamma} |f(x)| \sum_{y \in \Lambda_n \setminus \Lambda_m} F_a(d(x, y)) \sum_{\substack{X \subset \Gamma: \\ y \in X}} \int_{\mathbb{C}^X} |z_y| |\mu_X|(dz) \\
& \leq M c_a e^{(v_a + c_a \kappa_a C_a^2)(t-s)} \sum_{x \in \Gamma} |f(x)| \sum_{y \in \Lambda_n \setminus \Lambda_m} F_a(d(x, y))
\end{aligned}$$

follows readily from Theorem 4.3 and assumption (5.1). For  $f \in \ell^1(\Gamma)$  and fixed  $t$ , the upper estimate above goes to zero as  $n, m \rightarrow \infty$ . In fact, the convergence is uniform for  $t \in [-T, T]$ . This proves (5.2).

By an  $\epsilon/3$  argument, similar to what is done at the end of Section 2, weak continuity follows since we know it holds for the finite volume dynamics. This completes the proof of Theorem 5.1.  $\square$

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